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A Note on Casson-Gordon's Invariants

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In [1], Casson-Gordon showed that there exist knots which are algebraically null-cobordant but not null cobordant. Following them, we will observe that those knots in [1] are not cobordant to the knots whose Alexander polynomials are trivial.

We work in the smooth category and, unless otherwise stated, all manifolds are compact and oriented and homology is with integral coefficients.

1. Casson-Gordon's invariants

We will review the definitions of their invariants by Casson-Gordon; in fact, they have given three definitions and we will observe the equality of them.

Let M be a closed manifold, and $\psi : H_1(M) \longrightarrow Z_m$ an epimorphism. Then ψ induces an m-fold cyclic covering $\tilde{M} \longrightarrow M$, with a cannonical generator, corresponding to $1 \in Z_m$, for the group of covering translations.

From LEMMA 2.2 in [1], there is an m-fold cyclic branched covering of 4-manifolds $\tilde{W} \longrightarrow W$, branched over a surface $F \subset \mathrm{int}\, W$ with inverse image $\tilde{F} \subset \mathrm{int}\, \tilde{W}$, such that $\partial(\tilde{W} \longrightarrow W) = (\tilde{M} \longrightarrow M)$ and such that the cannonical covering translation τ of $\tilde{W}$ which induces rotation through $2\pi/m$ on the fibres of the normal bundle of $\tilde{F}$ restricts on $\partial\tilde{W}$ to the cannonical covering translation of $\tilde{M}$ determined by ψ. The intersection form on $H_2(\tilde{W})$ extends naturally to a Hermitian form $\cdot$ on $H = H_2(\tilde{W}) \otimes \mathbb{C}$. Let $\tau = \tau_*$: $H \longrightarrow H$ be the automorphism induced by τ. Then τ is an isometry on $(H, \cdot)$, and $\tau^m = \mathrm{id}$. Write $\omega = e^{\frac{2\pi i}{m}}$, and let E_r be the

ω^r-eigenspace of τ, $0 \le r < m$. Then $(H,\cdot)$ decomposes as an orthogonal direct sum $E_0 \oplus \cdots \oplus E_{m-1}$. Let $\varepsilon_r(\tilde{W})$ be the signature of the restriction of $\cdot$ to E_r. Now define for $0 < r < m$, the rational number $\sigma_r(M,\psi)$,

$$\sigma_r(M,\psi) = \text{sign } W - \varepsilon_r(\tilde{W}) - \frac{2[F]^2 r(m-r)}{m^2}$$

where $[F]^2$ denotesthe self-intersection number of the homology class represented by F. Well-definedness of $\sigma_r(M,\psi)$ follows from LEMMA 2.1 in [1] and Novikov additivity.

We will give other descriptions for $\sigma_r(M,\psi)$.

(1) (c.f. [2] or [3])

For given (M,ψ), from the finiteness of $\Omega_3(K(Z_m,1))$, there exists an m-fold covering of 4-manifolds $\tilde{W}_1 \longrightarrow W_1$ such that $\partial(\tilde{W}_1 \longrightarrow W_1) = k(\tilde{M} \longrightarrow M)$ for some integer $k \neq 0$. Let τ_1 be the generator of Z_m-action on $\tilde{W}_1$ which restricts on each component of $\partial\tilde{W}_1$ to the cannonical covering translation on $\tilde{M}$ determined by ψ. Now define for $0 < r < m$,

$$\sigma_r^1(M,\psi) = \frac{1}{k}(\text{sign } W_1 - \varepsilon_r(\tilde{W}_1))$$

where $\varepsilon_r(\tilde{W}_1)$ is the signature of the restriction of $\cdot$ to the ω^r-eigenspace of τ_1. Then from LEMMA 2.1 in [1] and Novikov additivity, it is easily seen that $\sigma_r = \sigma_r^1$.

(2) (c.f. [1])

For given (M,ψ), suppose that for some integer n, there is an mn-fold cyclic covering of 4-manifolds $\tilde{W}_2 \longrightarrow W_2$ such that $\partial(\tilde{W}_2 \longrightarrow W_2) = (\text{n-copies of } \tilde{M}) \longrightarrow M$ and, for some covering translation τ_0 which generates the covering translation group of $\tilde{W}_2$, $\tau_2 = \tau_0^n$ restricts on each component of $\partial\tilde{W}_2$ to the cannonical covering translation of $\tilde{M}$ determined by ψ. Now define for $0 < r < m$,

$$\sigma_r^2(M,\psi) = \text{sign } W_2 - \frac{1}{n}\varepsilon_r(\tilde{W}_2)$$

where $\varepsilon_r(\tilde{W}_2)$ is the signature of the restriction of $\cdot$ to the ω^r-eigenspace of τ_2.

We observe that $\sigma_r^1 = \sigma_r^2$ in the following. Now consider Z_m-action on $\tilde{W}_2$ generated by τ_0^n and denote the quotient space by $\tilde{W}_2/\tau_0^n$. Then $\tilde{W}_2 \longrightarrow \tilde{W}_2/\tau_0^n$ is the m-fold cyclic covering such that $\partial(\tilde{W}_2 \longrightarrow \tilde{W}_2/\tau_0^n) = n(\tilde{M} \to M)$. From this,

$$\sigma_r^1(M,\psi) = \frac{1}{n}(\mathrm{sign}(\tilde{W}_2/\tau_0^n) - \varepsilon_r(\tilde{W}_2))$$

Therfore it suffices to see that $\mathrm{sign}(\tilde{W}_2/\tau_0^n) = n\ \mathrm{sign}\ W_2$. To do so, let τ_0/τ_0^n denote the homeomorphism induced from τ_0 on $\tilde{W}_2/\tau_0^n$. Then Z_n-action generated by τ_0/τ_0^n is induced on $\tilde{W}_2/\tau_0^n$ and

$$\tilde{W}_2/\tau_0^n \longrightarrow W_2 = \tilde{W}_2/\tau_0^n \Big/ \tau_0/\tau_0^n$$

is an n-fold cyclic covering such that $\partial(\tilde{W}_2/\tau_0^n \longrightarrow W_2) = $ (n-copies of $M) \longrightarrow M$. Let W_3 be a 4-manifold with $\partial W_3 = M$. Pasting the copies of W_3 along M's and extending Z_n-action obiously, we obtain the n-fold cyclic covering $\tilde{X} \longrightarrow X$, where $X = W_2 \cup W_3$ and $\tilde{X} = \tilde{W}_2/\tau_0^n \cup$(n-copies of W_3). Now using Lemma 2.1, Novikov additivity and the equality $\sum_{i=0}^{n-1} \varepsilon_i(\tilde{X}) = \mathrm{sign}\ \tilde{X}$ (where $\varepsilon_i(\tilde{X})$ is the signature of the restiction of $\cdot$ to $(e^{\frac{2\pi i}{n}})^r$-eigenspace of τ_0/τ_0^n),

$$n\ \mathrm{sign}\ W_3 + \mathrm{sign}(\tilde{W}_2/\tau_0^n) = \mathrm{sign}\ \tilde{X} = \sum_{i=0}^{n-1} \varepsilon_i(\tilde{X}) = n\ \mathrm{sign}\ X$$

$$= n(\mathrm{sign}\ W_3 + \mathrm{sign}\ W_2)$$

This completes the proof.

2. Statement of the result

Let K be a knot in S^3. Fix a prime q, and let M_n denote the q^n-fold branched cyclic cover of (S^3,K), $n = 1,2,\cdots$. Then $H_*(M_n;Q) = H_*(S^3;Q)$. Suppose that we have an epimorphism ψ: $H_1(M_1) \longrightarrow Z_m$. Then the branched covering projection $M_n \longrightarrow M_1$ induces a surjection $H_1(M_n) \longrightarrow H_1(M_1)$. Composition with ψ then defines epimorphism ψ_n: $H_1(M_n) \longrightarrow Z_m$ for all n.

Theorem. Suppose that K is cobordant to the knot K' with trivial Alexander polynomial. Then there is a constant c, and a subgroup G of $H_1(M_1)$ with $|G|^2=|H_1(M_1)|$, such that if m is a prime power and $\psi : H_1(M_1) \longrightarrow Z_m$ is an epimorphism satisfying $\psi(G) = 0$, then $|\sigma_r(M_n,\psi_n)| < c$ for all n.

Let K_k $(k \in Z)$ denote the k-twisted double of the unknot. Then, from Theorem and the proof of THEOREM 5.1 in [1], we obtain

Corollary. K_k is cobordant to the knot with trivial Alexander polynomial only if $k = 0, 2$.

3. Lemmas. ~~Let W be a Q-homology cobordism between Q-homology~~

Lemma 1. Let W be a Q-homology cobordism between Q-homology 3-sphere M and Z-homology 3-sphere M'. If the image of the map $H_1(M) \longrightarrow H_1(W)$ induced from inclusion has order ℓ, then $H_1(M)$ has ℓ^2.

This is an easy generalization of LEMMA 4.1 in [1].

Lemma 2. Let M_n' denote the q^n-fold branched cyclic cover of (S^3,K'), where K' has trivial Alexander polynomial. Then, for all n,

(1) M_n' is Z-homology 3-sphere, and

(2) M_n' bounds simply connected 4-manifold with index zero.

4. Proof of THEOREM.

By hyphothesis, there is a smooth submanifold T of $S^3 \times I$ homeomorphic to $S^1 \times I$, such that $T \cap S^3 \times 0 = K$, $T \cap S^3 \times 1 = K'$. For fixed prime q, W_n denotes the q^n-fold branched cyclic cover of $(S^3 \times I, T)$ and M_n (resp. M_n') denotes the q^n-fold branched cyclic cover of (S^3,K) (resp. (S^3,K')). Then, W_n gives the Q-homology cobordism between M_n and M_n'. Let $i_{n*}: H_1(M_n) \longrightarrow$

$H_1(W_n)$ be induced by inclusion, and let $G = \mathrm{Ker}\, i_{1*}$. By Lemma 1. and Lemma 2(1), $|G|^2=|H_1(M_1)|$.

Suppose that $m = p^a$, where p is prime. By similar argument to that of THEOREM 4.1 in [1], we come to the following situation;

(1) we set $d_n = \dim H_1(W_n,Z_p)$, $n = 1,2\cdots$,

(2) W'_n is obtained from W_n, by doing surgery on d_n-1 circles in interior W_n,

(3) $H_1(W'_n,Z_p)$ is cyclic,

and

(4) the following diagram is commutativf for all n ;

$$\begin{tikzcd} H_1M_n \arrow[r, "i_{n*}"] \arrow[d, "\psi_n"'] & H_1W'_n \arrow[d, "\Psi'_n"] \\ Z_{p^a} \arrow[r] & Z_{p^b} \end{tikzcd}$$

where i'_n is inclusion, Ψ'_n is surjective and $Z_{p^a} \longrightarrow Z_{p^b}$ is multiplication by p^{b-a}. Let $\tilde{W}'_n \longrightarrow W'_n$ be the p^b-fold cyclic covering induced by Ψ'_n; then $\partial(\tilde{W}'_n \longrightarrow \tilde{W}'_n)$ consists of (p^{b-a}-copies of $\tilde{M}_n) \to M_n$ and (p^b-copies of $M'_n) \to M'_n$, where $\tilde{M}_n \longrightarrow M_n$ is the covering induced by ψ_n and $M'_n \to M'_n$ is the trivial covering. From Lemma 2(2), M'_n bounds simply connected 4-manifold W''_n with index zero. Pasting copies of W''_n to $\tilde{W}'_n \to W'_n$ along M'_n's, we obtain the covering $\tilde{V}_n \to V_n$, where $V_n = W'_n \cup W''_n$.

Using this covering $\tilde{V}_n \to V_n$, we can give an estimate for $\sigma_r(M_n,\psi_n)$. From the construction, sign $V_n = 0$ and $\varepsilon_r(\tilde{V}_n) = \varepsilon_r(\tilde{W}'_n) + \varepsilon_r(W''_n \cup \cdots \cup W''_n)$. By linear algebraic argument, $\varepsilon_r(W''_n \cup \cdots \cup W''_n) = 0$; that is, $\varepsilon_r(\tilde{V}_n) = \varepsilon_r(\tilde{W}'_n)$. Therefore if $\tilde{X}$ denotes the infinite cyclic cover of $S^3 \times I - T$, from similar caluculation on dim. $H_2(\tilde{W}'_n,Q)$ to that of [1], we obtain the estimate

$$|\sigma_r(M_n,\psi_n)| < |G|(2d-1) + 1$$

where d is equal to $\dim. H_1(\tilde{X},Z_p)$. This completes the proof.

Appendix

Non-Ribbon Pretzel Knot

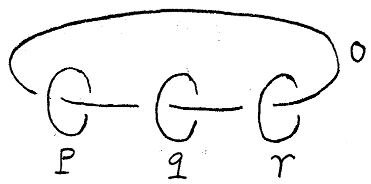
In [2], Casson-Gordon gave the examples that are algebraically slice but not ribbon among 2-bridge knots. Our example is a direct consequence of Hsiang-Szczarba's result in [4] on the representability of 2-dimensional homology class of 4-manifold by an embedded 2-sphere. We need some lemmas.

Lemma 1. Let $K \subset S^3$ be a ribbon knot and Σ_2 be the 2-fold branched cover of S^3 branched over K. Then Σ_2 bounds a Q-acyclic 4-manifold V such that the homomorphism $i_*: \pi_1(\partial V) \to \pi_1(V)$ induced by inclusion is onto.

See [2].

Lemma 2. Let $K = K(p,q,r)$ be a Pretzel knot of type (p,q,r). where p,q,r are odd integers. Then its 2-fold branched cover is the 3-manifold with the following surgery diagram.

See [5].



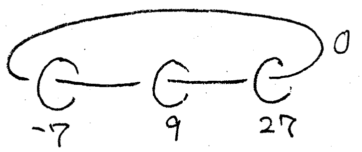
Lemma 3. Let $K = K(p,q,r)$ be as in Lemma 2. Then K is algebraically slice if and only if $|pq + qr + rp| =$ square.

This follows from the direct calculation of Seifert matrix.

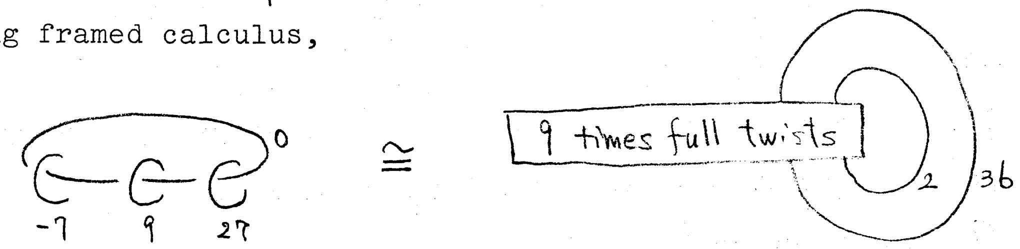
Example. $K = K(-7,9,27)$ is algebraically slice but not ribbon.

Proof. From Lemma 3, K is algebraically slice.
Assume that K is ribbon. Then Σ_2, 2-fold branched cover

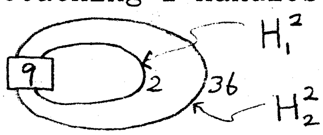
of (S^3,K), bounds V as in Lemma 1. By Lemma 2, Σ_2 is



Using framed calculus,



That is, Σ_2 bounds 4-manifold W which is obtained from the 4-ball by attaching 2-handles H_1^2 and H_2^2 following the framed link



We construct 4-manifold X by pasting V and W along their boundaries. Then X is simply connected closed 4-manifold whose intersection module is unimodular, even, positive definite and with rank 2. Therefore, from the classification theorem of the integral quadratic modules and homotopy classification theorem of simply connected 4-manifolds (see [6]), X is homotopy equivalent to $S^2 \times S^2$. From this, we can choose a symplectic basis for $H_2(X)$, that is, $\{\alpha,\beta\}$ is a basis for $H_2(X)$ such that $\alpha\cdot\alpha = \beta\cdot\beta = 0$, $\alpha\cdot\beta = 1$. We denote by γ_1 and γ_2 the homology classes corresponding to the cores of the 2-handles H_1^2 and H_2^2. Write γ_1 and γ_2 by $\{\alpha,\beta\}$;

$$\begin{aligned} \gamma_1 &= a\alpha + b\beta \\ \gamma_2 &= c\alpha + d\beta \end{aligned} \qquad (a,b,c,d \in Z)$$

Then $\gamma_1\cdot\gamma_1 = 2ab = 2$, $\gamma_2\cdot\gamma_2 = 2cd = 36$ and $\gamma_1\cdot\gamma_2 = ad + bc = 9$. Thus γ_2 must represent $\pm(3\alpha + 6\beta)$ or $\pm(6\alpha + 3\beta)$. Therefore γ_2 is non-primitive and represented by embedded S^2. This contradicts to THEOREM 1.1 in [4].

References

[1] A.J. Casson and C.McA. Gordon, On Slice Knots In Dimension Three, Proc. of Sympo. in Pure Math. Vol.32, 1978, 39-53.

[2] ______________, Cobordism of classical knots, mimeographed notes, Orsay, 1975.

[3] C.MaC. Gordon, Some aspects of classical knot theory, Lecture Notes in Mathematics. Vol.685, 1978, 1-60.

[4] W.C. Hsiang and R.H. Szczarba, On embedding surfaces in four-manifolds, Proc. Sympos. Pure Math. Vol.22, 1971, 97-103.

[5] J.M. Montesinos, Variedades de Seifert que son recubridores ciclicos de doh hojas, Bol. Soc. Mat. Mexicana (2) 18, 1973, 1-32.

[6] J.Milnor and D. Husemoller, Symmetric Billnear Forms, Springer-Verlag, New York, 1973.